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Toroidal Constraints for Two-Point Localization under High Outlier Ratios
Supplementary Material Document

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Abstract

This paper provides supplementary material for our submission. Herein, we offer additional derivation details for the geometric solver described in the main paper. First, we show that the original cost can be parameterized to yield a minimum of 24 solutions. Then, we detail the steps to find the factors of the final approximate solution.

A. Solutions to the Full Cost

In this section we give more details on the derivation of a solver that can find the position of the camera on $\mathbb{T}^2$. As opposed to the approximate solver (which is detailed in Section B), the solver described in this section aims to get the location on $\mathbb{T}^2$ globally and exactly by minimizing an energy function. $E(u, v)$ in Eq. 2[1] is the energy we seek to minimize, *i.e.*

$$E(u,v) = \measuredangle\left(P_0(u,v), q_0\right)^2 + \measuredangle\left(P_1(u,v), q_1\right)^2 , \tag{A.1}$$

where u and v are the toroidal coordinates (*i.e.* they are angles that parameterize a point on the surface of the torus), and where q_0 and q_1 are the triangulation rays. Furthermore, we define P_i as the vector from the i-th 3D point to the camera center $C \in \mathbb{T}^2$, *i.e.*

$$P_i(u,v) = C(u,v) - p_i , \tag{A.2}$$

where p_i is the position of the i-th 3D point matched. Note that $p_0 = -p_1$ and that $r^2 = R^2 + \|p_i\|^2$, where R and r are the major and minor radii of the torus (*c.f.* Section 3.2 - Toroidal Constraints).

By minimizing E, we ensure that the position on $\mathbb{T}^2$ is optimal w.r.t. our assumptions (*c.f.* Section 3.3 - Triangulation-Ray Constraints). These imply that we are to minimize the sum of the squares of the *angular* distances. We define thus

$$\sum_{i=0,1} \measuredangle\left(P_i(u,v), q_i\right)^2 = \sum_{i=0,1} \arcsin\left(\frac{\|P_i(u,v) \times q_i\|}{\|P_i(u,v)\|}\right)^2 . \tag{A.3}$$

Here we use the notation $C(u, v)$ since the camera position C can be parameterized as in Eq. 1. However, in order to find u and v that minimize Eq. A.3 we choose, for simplicity, to instead minimize the square of the sine of the angles between vectors, *i.e.*

$$(u^*, v^*) = \arg\min_{u,v} \sum_{i=0,1} \left(\frac{\|P_i(u,v) \times q_i\|}{\|P_i(u,v)\|}\right)^2 = \arg\min_{u,v} \sum_{i=0,1} f_i\left(C(u,v)\right)^2 . \tag{A.4}$$

Since we need to find all meaningful minima of Eq. A.4 (*c.f.* Section 3.4 - A Geometric Solver), we seek a closed for for all its stationary points. We explored two different parameterizations of the problem for this: trigonometric and Lagrangian.

[1] All numeric references refer to the main paper.

A.1. Lagrangian Formulation

In the Lagrangian formulation, we let the 3D position of the camera $C = (x, y, z)^T \in \mathbb{R}^3$ be our unknown and then enforce that it lies on $\mathbb{T}^2$ using Lagrangian multipliers. Implicitly, the surface of a torus is

$$\begin{aligned} g(x,y,z) &= (x^2+y^2+z^2+R^2-r^2)^2 - 4R^2(x^2+y^2) \\ &= (\|C\|^2 - \|p_0\|^2)^2 - 4R^2(\|C\|^2 - z^2)\,. \end{aligned} \tag{A.5}$$

Using this we can express the Lagrangian as

$$\mathcal{L}(x,y,z,\lambda) = \sum_{i=0,1} f_i\,(x,y,z)^2 - \lambda g(x,y,z)\,. \tag{A.6}$$

The stationary points are when $\nabla\mathcal{L}(x,y,z,\lambda) = 0$. Immediately, one might see that this gradient will result in a system of four equations in (x, y, z, λ). Here we detail the derivation of these four equations:

$$\nabla_{xyz}\, f_i^2 = \frac{\partial f_i(x,y,z)^2}{\partial(x,y,z)} = \frac{1}{\|P_i\|^4\|q_i\|^2}\left(\|P_i\times q_i\|^2 P_i - \|P_i\|^2 q_i \times (P_i \times q_i)\right) \tag{A.7a}$$

$$\nabla_{xyz}\, g = 2(\|C\|^2 - \|p_0\|^2)C - 4R^2(x,\ y,\ 0)^\top \tag{A.7b}$$

$$\nabla\mathcal{L}(x,y,z,\lambda) = \begin{bmatrix} \sum_{i=0,1} \nabla_{xyz} f_i^2 + \lambda\nabla_{xyz}\, g \\ g(x,y,z) \end{bmatrix}_{4\times 1} = 0_{4\times 1} \tag{A.7c}$$

To analyze the number of solutions for the system of equations, we chose to employ a Gröbner basis technique. To this end, we need to have only polynomial equations in our system. After some algebraic manipulation, we arrive at three polynomials of total degree 12 and one polynomial of total degree 4. Using Macaulay2 [1] we find that, under this formulation, there are up to 47 real solutions. Detailed in the next subsection, we note that we can reduce this to 24 solution by using a different parameterization.

A.2. Trigonometric Formulation

A simpler solution can be obtained if we use toroidal coordinates, *i.e.* if we use the angles u and v to express C (*c.f.* Eq. 1, but repeated here):

$$C(u,v) = \begin{bmatrix} (R + r\cos v)\cos u \\ (R + r\cos v)\sin u \\ r\sin v \end{bmatrix} \in \mathbb{T}^2\,. \tag{A.8}$$

Using this parameterization we can avoid the use of Lagrangian multipliers, which introduce an auxiliary variable (*i.e.* λ) and an extra equation. This lets us write the objective function as simply

$$E(u,v) = \sum_{i=0,1} f_i\,(u,v)^2\,. \tag{A.9}$$

However, we need to modify $\nabla_{xyz} f_i^2$ (*c.f.* Eq. A.7a) to operate in this coordinates. This requires the Jacobian of $C(u, v)$ w.r.t. u and v,

$$J_{uv} = \begin{bmatrix} -\sin u(R + r\cos v) & -r\cos u \sin v \\ \cos u(R + r\cos v) & -r\sin u\sin v \\ 0 & r\cos v \end{bmatrix}\,. \tag{A.10}$$

And thus, the gradient parametric in u and v is

$$\nabla_{uv}\, f_i^2 = \nabla_{xyz}\, f_i^2 \cdot J_{uv}\,. \tag{A.11}$$

Once more, getting the final gradient parametric in u and v and setting it to zero gives us our system of equations:

$$\sum_{i=0,1} \nabla_{uv} f_i^2 = 0_{2\times 1}\,, \tag{A.12}$$

which are two equations in u and v. This time, however, we cannot convert Eq. A.12 into a polynomial system since it is a function of sines and cosines of u and v. To do this we have two choices. The first choice is the tangent half-angle substitution (also known as the Weierstrass substitution) where

$$\sin\alpha = \frac{2t}{1+t^2} \quad \text{and} \quad \cos\alpha = \frac{1-t^2}{1+t^2}\,, \tag{A.13}$$

where $t = \tan\frac{\alpha}{2}$. This adds no new equations but raises the degree of the equation significantly, since for each trigonometric term we introduce a rational quadratic polynomial. The second choice is using the more straightforward "circular"' substitution

$$\sin\alpha = a \quad \text{and} \quad \cos\alpha = b\,, \tag{A.14}$$

which does not raise the degree of the equations. However, this substitution duplicates the number of variables and adds a new equation per angle substituted, *i.e.* $a^2 + b^2 = 1$. Once again we can use Macaulay2 to get the number of solutions that such parameterizations would yield. In the case of the tangent half-angle, we get up to 32 real solutions, whereas the circular substitution will generate up to 24 real solutions.

B. Derivation Details of the Approximate Solver

Our aim is to find a closed-form solution to Eq. 6. For this we want to find all stationary points of $\hat{E}$. In Section 3.4 (A Geometric Solver) we claim that there are 12 such stationary points. To arrive at this number, we again follow a Gröbner basis method. Expanding Eq. 7 and discarding the common denominator we get

$$\begin{aligned}
&\frac{\partial \hat{E}}{\partial v} = (\mu_0\ \rho_0\ \phi_0)\ \xi_1 + (\mu_1\ \rho_1\ \phi_1)\ \xi_0 = 0\,, \quad \text{where}\\
&\mu_i = \left(r s_i^2 \cos v + r\cos v + R s_i^2 + R\right) \quad\ \rho_i = (p_{iz}\sin v + r + R\cos v)\\
&\phi_i = (r s_i \cos v - r\sin v + R s_i - p_{iz}) \quad\ \xi_i = (s_i p_{iz} + r s_i \sin v + r\cos v + R)^{\,3}\,,
\end{aligned} \tag{B.1}$$

from which we can more easily see that the maximum degree of $\frac{\partial \hat{E}}{\partial v}$ is $\cos{(v)}^6$. Since we are already dealing with an equation of quite high total degree (*i.e.* trigonometric monomials of total degree 6), we opt to make the substitution in Eq. A.14 to continue our analysis:

$$\cos v = c_v \quad \sin v = s_v\,. \tag{B.2}$$

Furthermore, if we were to use the half-tangent angle substitution we would end up with up to 24 solutions. As we will see, by using this substitution we can get a solution with only 12 roots. Using Eq. B.2 in Eq. B.1 we get

$$\begin{aligned}
&k_0 c_v^6 + k_1 c_v^5 s_v + k_2 c_v^5 + k_3 c_v^4 s_v^2 + k_4 c_v^4 s_v + k_5 c_v^4 + k_6 c_v^3 s_v^3 + k_7 c_v^3 s_v^2 + k_8 c_v^3 s_v + k_9 c_v^3 +\\
&k_1 0 c_v^2 s_v^4 + k_1 1 c_v^2 s_v^3 + k_1 2 c_v^2 s_v + k_1 3 c_v^2 + k_1 4 c_v s_v^5 + k_1 5 c_v s_v^4 + k_1 6 c_v s_v^3 + k_1 7 c_v s_v^2 + k_1 8 c_v s_v +\\
&k_1 9 c_v + k_2 0 s_v^5 + k_2 1 s_v^4 + k_2 2 s_v^3 + k_2 3 s_v^2 + k_2 4 s_v + k_2 5 = 0\\
&c_v^2 + s_v^2 = 1\ ,
\end{aligned} \tag{B.3}$$

where $k_i = k_i(R, r, p_{0z}, p_{1z}, s_0, s_1)$ for all $i = 0..25$ are constants that depend on the torus parameters and the triangulation ray directions. After inspecting Eq. B.3 with Macaulay2, we find that we can have up to 12 real solutions. As mentioned in Section 3.4 (A Geometric Solver), we observe that 6 of the 12 solutions are repeated and invalid for our geometric requirements. Indeed, the terms ρ_0 and ξ_1 vanish simultaneously if $r\cos v = -R$, *i.e.* if we plug in an invalid solution then Eq. B.1 will be zero. This solution corresponds to having the camera position along the circle coincident with either p_0 or p_1. At this point the angle between q_0 or q_1 is undefined.

Thus, we must find a way to solve Eq. B.3 without resorting to a method that will necessarily solve for all 12 solutions, such as a Gröbner basis approach. If we expand using the tangent half-angle substitution, the equations can more easily be simplified. Indeed, we can obtain a univariate polynomial in t for which we can efficiently find all real roots using a Sturm-bracketing plus polishing method [2]. However, we can instead manipulate this polynomial in order to factor out unwanted terms symbolically. This would result in a much simpler method and we can easily identify and discard the roots that we know in advance will produce spurious solutions. Namely, under this substitution the invalid roots come from the factor $(r + R + t^2(R - r))$, where $t = \tan{(v/2)}$.

Either manually or using a Computer Algebra Program, one can show that one may factor Eq. B.1 to Eq. 8a, *i.e.*:

$$
\begin{aligned}
0 =&(r + R + t^2(R - r))^3 \cdot (\lambda_1 + \lambda_2 t + \lambda_3 t^2)\cdot \\
&((s_1 + s_0 + t(4s_0s_1 - 2) - t^2(s_0 + s1)) \cdot (f_c(t))
\end{aligned}
\tag{B.4a}
$$

where

$$
\begin{aligned}
\lambda_1 &= \kappa - \tau \\
\lambda_2 &= 2r(s_0 + s_1)(p_{1z}(s_1 - s_0) + Rs_0s_1 + R) \\
\lambda_3 &= \kappa + \tau
\end{aligned}
\tag{B.4b}
$$

and

$$
\begin{aligned}
\kappa = & R^2\left(s_0^2\left(s_1^2 - 1\right) + 4s_0s_1 - s_1^2 + 1\right) + \\
& r^2\left(s_0^2\left(2s_1^2 + 1\right) + 2s_0s_1 + s_1^2 + 2\right) - \\
& 2Rp_{1z}\left(s_0^2 s_1 - s_0 s_1^2 + s_0 - s_1\right) \\
\tau = & r(s_0s_1 - 1)(p_{1z}(s_1 - s_0) + Rs_0s_1 + R)\,.
\end{aligned}
\tag{B.4c}
$$

Finally, in Eq. B.4a the factor

$$
f_c = (r + R + t^2(R + r))
\tag{B.5}
$$

has only complex roots

$$
t = \pm\frac{\sqrt{-(R + r)}}{\sqrt{R + r}}\,,
\tag{B.6}
$$

since $r > 0$ and $R > 0$.

References

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